MMAT 5000: Analysis I (2016 1st term)

1 Basic Definitions

Throughout the note, we use the following notation:

(i) $\mathbb{R}$ = the set of all real numbers.

(ii) $\mathbb{C}$ = the set of all complex numbers.

(iii) $\mathbb{Q}$ = the set of all rational numbers.

(iv) $\mathbb{N}$ = the set of all natural numbers.

Definition 1.1 Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if it satisfies the following conditions.

(i) $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) (Symmetric property) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, the pair (X, d) is called a metric space.

Example 1.2 :

(i) For $x, y \in \mathbb{R}$, put $d(x, y) = |x - y|$. Then d is a metric on $\mathbb{R}$ and d is called the usual metric on $\mathbb{R}$.

(ii) For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, define:
$d_\infty(x, y) = \max(|x_1 - x_2|, |y_1 - y_2|)$;
$d_1(x, y) = |x_1 - x_2| + |y_1 - y_2|$;
$d_2(x, y) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$. Then all are metrics on $\mathbb{R}^2$.

(iii) Let X be any non-empty set. For $x, y \in X$, let $d(x, y) = 0$ if $x = y$; otherwise, $d(x, y) = 1$. Then d is a metric on X. In this case, d is called the discrete metric on X and (X, d) is called a discrete metric space.

(iv) Fix a prime number p. For $\frac{a}{b} \in \mathbb{Q}$, define $|\frac{a}{b}|_p = p^{-v}$ if $\frac{a}{b} = p^v \frac{a'}{b'}$ where $v \in \mathbb{Z}$ and $p \nmid a'b'$. If we put $d_p(x, y) = |x - y|_p$ for $x, y \in \mathbb{Q}$, then d_p is a metric on $\mathbb{Q}$. Furthermore, d_p satisfies the strong triangle inequality, i.e.,

$$d_p(x, y) \leq \max(d_p(x, z), d_p(z, y))$$

for all $x, y, z \in \mathbb{Q}$.

Definition 1.3 Let V be a vector space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is called a norm on V if it satisfies the following conditions.

(i) $\|x\| \geq 0$ for all $x \in V$.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(iii) (Triangle inequality) $\|x - y\| \leq \|x - z\| + \|z - y\|$ for all $x, y, z \in V$.

In this case, the pair $(V, \|\cdot\|)$ is called a normed space.

Proposition 1.4 *Let $(V, \|\cdot\|)$ be a normed space. If we put $d(x, y) = \|x - y\|$ for $x, y \in V$, then d is a metric on V. Consequently, every normed space is a metric space.*

Remark 1.5 Let V be a vector space. Notice that the discrete metric d on V must not be induced by a norm, i.e., we cannot find a norm $\|\cdot\|$ on V such that $d(x, y) = \|x - y\|$ for $x, y \in V$.

Example 1.6 The following are important examples of normed spaces.

(i) Let $\ell^\infty = \{(x_n) : x_n \in \mathbb{C},\ n = 1, 2...;\ \sup |x_n| < \infty\}$ and $c_0 = \{(x_n) \in \ell^\infty :\ \lim |x_n| = 0\}$. Put $\|(x_n)\|_\infty = \sup |x_n|$.

(ii) Let $\ell^1 = \{(x_n) : x_n \in \mathbb{C},\ n = 1, 2...;\ \sum_{n=1}^\infty |x_n| < \infty\}$. Put $\|(x_n)\|_1 = \sum_{n=1}^\infty |x_n|$.

(iii) Let $\ell^2 = \{(x_n) : x_n \in \mathbb{C},\ n = 1, 2...;\ \sum_{n=1}^\infty |x_n|^2 < \infty\}$. Put $\|(x_n)\|_2 = \sqrt{\sum_{n=1}^\infty |x_n|^2}$.

Exercise 1.7 :

(1) Let (X, d) be a metric space. Define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

for $x, y \in X$. Show that ρ is also a metric on X.

(2) Let (X, d_X) and (Y, d_Y) be the metric spaces. Define

$$\rho((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$

for $x, x' \in X$ and y, y' in Y. Show that ρ is a metric on the product space $X \times Y = \{(x, y) :\ x \in X; y \in Y\}$.

(3) Let (X, d) be a metric space and let A be a subset of X. We say that A is bounded if there is $M > 0$ such that $d(a, a') \leq M$ for all a, a' in A.
Show that if $A_1, ..., A_N (N < \infty)$ all are bounded subsets of X, show that $A_1 \cup \cdots \cdots \cup A_N$ is also a bounded subset of X.

2 Convergent Sequences

Throughout this section, (X, d) will denote a metric space.
For $a \in X$ and $r > 0$, put
$B(a, r) = \{x \in X : d(a, x) < r\}$, called the *open ball* with center a of radius r;
$\overline{B}(a, r) = \{x \in X : d(a, x) \leq r\}$, called the *closed ball* with center a of radius r.

Recall that a sequence on X is a function $f : \{1, 2, ..\} \to X$. Write $f(n) = x_n \in X$. Also, if (n_k) is a sequence of positive integers with $n_1 < n_2 < n_3 < \cdots\cdots$, then we call (x_{n_k}) a subsequence of (x_n).

Definition 2.1 A sequence (x_n) is said to be convergent in X if there is an element $a \in X$ such that $d(a, x_n) \to 0$ as $n \to \infty$, *i.e.*, it satisfies the following condition.

For any $\varepsilon > 0$, there is a positive integer N such that $x_n \in B(a, \varepsilon)$ for all $n \geq N$.

In this case, a is called a limit of the sequence (x_n). Also (x_n) is said to be divergent if it is not convergent

Proposition 2.2 *If (x_n) is a convergent sequence in X, then its limit is unique. Now we can write* $\lim x_n$ *for the limit of (x_n).*

Proof: Suppose that a and b both are the limits of (x_n) with $a \neq b$ in X. Then $d(a, b) > 0$. Choose $0 < 2\varepsilon < d(a, b)$. By the definition of limit, we can find the integers N_1 and N_2 such that $d(a, x_n) < \varepsilon$ for all $n \geq N_1$ and $d(b, x_n) < \varepsilon$ for all $n \geq N_2$. Now if we take $N \geq \max(N_1, N_2)$, then we have

$$d(a, x_N) < \varepsilon; \text{ and } d(b, x_N) < \varepsilon.$$

Hence we have

$$d(a, b) \leq d(a, x_N) + d(x_N, b) < 2\varepsilon < d(a, b).$$

It leads to a contradiction. □

Example 2.3 :

(i) If we let $(\mathbb{R}, d)$ be the usual metric space and let $x_n = 1/n$, then (x_n) is a convergent sequence in $\mathbb{R}$.

(ii) If we let $X = (0, 1]$ and d is the metric induced by the usual metric on $\mathbb{R}$, then the sequence $(1/n)$ is divergent in $(0, 1]$. In fact, if $(1/n)$ converges to an element $a \in (0, 1]$, then $\lim 1/n = a$ in $\mathbb{R}$. Then by the uniqueness of limit (see Proposition 2.2), we have $a = 0$. It leads to a contradiction.

Definition 2.4 Let A be a subset of X. A point $a \in X$ is said to be a limit point of A if for any $r > 0$, we have

$$(B(a, r) \setminus \{a\}) \cap A \neq \emptyset$$

i.e., for any $r > 0$, there is an element $z \in A$ such that $0 < d(a, z) < r$ (note: $z \neq a$ because $d(a, z) > 0$).

Put $D(A)$ the set of all limit points of A and $\overline{A} = A \cup D(A)$. Also the set $\overline{A}$ is called the closure of A.

Proposition 2.5 *Using the notation above, let $z \in X$. Then the following are equivalent.*

(i) $z \in \overline{A}$.

(ii) $B(z,r) \cap A \neq \emptyset$ *for all* $r > 0$.

(iii) There is a sequence $(x_n) \in A$ *such that* $\lim x_n = z$.

Moreover, if A and B are any subsets of X, then we have

(a) $\overline{\emptyset} = \emptyset$.

(b) $\overline{\overline{A}} = \overline{A}$.

(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Remark 2.6 (i) In general, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
For example, if we consider $X = \mathbb{R}$ and $A = (0,1); B = (1,2)$, then $A \cap B = \emptyset$ and $\overline{A} = [0,1], \overline{B} = [1,2]$. So, we have $\emptyset = \overline{A \cap B} \subsetneq \overline{A} \cap \overline{B} = \{1\}$.

(ii) Let $A_1, A_2, ...$ be an infinite sequence of subsets of X. In general, $\overline{\bigcup_{n=1}^{\infty} A_n} \neq \bigcup_{n=1}^{\infty} \overline{A_n}$.
For example, let $X = \mathbb{R}$ and $A_n = [0, 1 - \frac{1}{n})$. Then $\overline{\bigcup_{n=1}^{\infty} A_n} = [0,1]$ but $\bigcup_{n=1}^{\infty} \overline{A_n} = [0,1)$.

Example 2.7 (i) Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Then $D(\mathbb{Z}) = \emptyset$ and $\overline{A} = \mathbb{Z}$.

(ii) Let $X = \mathbb{R}$ and $A = (0,1]$. Then $D(A) = [0,1]$ and $\overline{A} = [0,1]$.

(iii) Let $X = (0,\infty)$ and $A = (0,1]$. Then $D(A) = (0,1]$ and $\overline{A} = (0,1]$.

(iv) Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then $D(A) = \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{R}$ (A is said to be dense in X if $\overline{A} = X$).

(v) Using the notation as in Example 1.6, we let

$$c_{00} = \{(x_n) \in \ell^{\infty} : \text{ there are only finitely many } x_n\ 's \neq 0\}.$$

Also c_{00} is endowed with the $\|\cdot\|_{\infty}$.
Then the set c_{00} is dense in c_0. In fact, if $v = (v_n) \in c_0$, then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|v_n| < \varepsilon$ for all $n \geq N$. Now we define $\xi = (\xi_n)$ by $\xi_n = v_n$ when $1 \leq n \leq N-1$ and $\xi_n = 0$ when $n \geq N$. Then $\xi \in c_{00}$ and $\|v - \xi\|_{\infty} = \sup_{n \geq N} |v_n| < \varepsilon$. So $v \in \overline{c_{00}}$.

Definition 2.8 A subset A of X is said to be closed in X if $\overline{A} = A (\Leftrightarrow D(A) \subseteq A)$.

Proposition 2.9 *A subset A of X is closed if and only if for an element $a \in X$ having a sequence (x_n) in A with* $\lim x_n = a$, *implies* $a \in A$.

Example 2.10 (i) Let $X = \mathbb{R}$. Then $\mathbb{Z}$ is a closed subset on $\mathbb{R}$ and $(0,1]$ is "Not" a closed subset of $\mathbb{R}$. However, if $X = (0,\infty)$, then $(0,1]$ is a closed subset of $(0,\infty)$.

So, the notion of "Closeness" depends on the choice of X.

(ii) Using the notation as in Examples 1.2 and 2.3, c_0 is a closed subspace of ℓ^∞ and c_{00} is not a closed subspace of ℓ^∞.
Claim : c_0 is closed in ℓ^∞.
By Proposition 2.9, we need to show that if $v \in \ell^\infty$ with a sequence (ξ_n) in c_0 such that $\lim_n \|\xi_n - v\|_\infty = 0$, then $v \in c_0$.
Now put $v = (v_j)_{j=1}^\infty$ and $\xi_n = (\xi_{n,j})_{j=1}^\infty$. Let $\varepsilon > 0$. Since $\lim_n \|\xi_n - v\|_\infty = 0$, there is a positive integer N such that $\|v - \xi_N\|_\infty < \varepsilon$. This implies that $|v_j - \xi_{N,j}| < \varepsilon$ for all $j \in \mathbb{N}$. On the other hand, there is a positive integer J such that $|\xi_{N,j}| < \varepsilon$ for all $j \geq J$ because $\xi_N \in c_0$. So, we have

$$|v_j| < |\xi_{N,j}| + \varepsilon < 2\varepsilon$$

for all $j \geq J$. Therefore, $v \in c_0$. The proof is finished.

Proposition 2.11 *Using the notation as before, we have the following assertions.*

(i) The whole set X and the empty set $\emptyset$ both are closed subsets of X.

(ii) If A and B are the closed subsets of X, then so is $A \cup B$.

(iii) If $(A_i)_{i\in I}$ is a family of closed subsets of X, then so is the intersection $\bigcap_{i\in I} A_i$.

(iv) The closure $\overline{A}$ of A is the smallest closed set containing A, that is, $\overline{A}$ is closed and if F is another closed set with $A \subseteq F$, then $\overline{A} \subseteq F$.

Remark 2.12 The assumption of the finite union of closed sets in Proposition 2.11 *(ii)* is essential. For example, consider $X = \mathbb{R}$ and $\bigcup_{n=2}^\infty [1/n, 1] = (0,1]$.

Exercise 2.13 Let A be a non-empty subset of X. A point $a \in X$ is called a boundary point of A if $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$ for all $r > 0$, where A^c denotes the complement of A in X. The set of all boundary points, write ∂A, of A is called the boundary of A.

(i) Find the boundaries of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$.

(ii) Let $X = (0,1) \cup (2,3)$. Find the boundary of the set $(0,1)$ in X.

(iii) Show that the boundary ∂A is a closed subset of X.

(iv) Show that $\overline{A} = A \cup \partial A$.

Definition 2.14 A subset V of X is said to be open in X if for each $z \in V$, there is $r > 0$ such that $B(z,r) \subseteq V$.

Remark 2.15 (i) The notion of open sets depends on the choice of X in which the sets are sitting. For example $(0,1]$ is not open in $\mathbb{R}$ but it is open in the set $(0,1] \cup [2,3]$.

(ii) A subset V of X can be an open and closed subset of X. For example, $(0,1]$ is open and closed subset of $(0,1] \cup [2,3]$.

(iii) A subset V can be neither closed nor open in X. For example, $(0,1]$ is neither closed nor open in $\mathbb{R}$.

Proposition 2.16 *We have the following assertions.*

(i) A subset V is open in X if and only if $X \setminus V$ is closed in X.

(ii) The empty set $\emptyset$ and the whole set X both are open.

(iii) If $\{V_i\}_{i\in I}$ is a family of open subsets of X, then the union $\bigcup_{i\in I} V_i$ is open in X.

(iv) For any finitely many $V_1, ..., V_N$ open subsets of X, we have $V_1 \cap \cdots \cap V_N$ is open in X. For example, $(0,1]$ is neither closed nor open in $\mathbb{R}$.

Exercise 2.17 (i) Let V be a subset of X. A point $z \in V$ is said to be an interior point of V if there is $r > 0$ such that $B(z,r) \subseteq V$. If we put $int(V)$ the set of all interior points of V, show that $int(V)$ is an open subset of X.

(ii) A metric d on X is said to be non-archimedean if it satisfies the strong triangle inequality, that is, $d(x,y) \le \max(d(x,z), d(z,y))$ for all x, y and $z \in X$ (see also Example 1.2 (iv)). Show that if d is a non-archimedean metric on X, then for every closed ball $\overline{B}(a,r) := \{x \in X : d(a,x) \le r\}$ is an open set in X.

3 Sequentially Compact Metric Spaces

Throughout this section, (X,d) always denotes a metric space. Let (x_n) be a sequence in X. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1,2,..\} \mapsto n_k \in \{1,2,...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \ge N$ and thus we have $n_k \ge N$ for all $k \ge K$.

Proposition 3.1 *Let (x_n) be a sequence in X. Then the following statements are equivalent.*

(i) (x_n) is convergent.

(ii) Any subsequence (x_{n_k}) of (x_n) converges to the same limit.

(iii) Any subsequence (x_{n_k}) of (x_n) is convergent.

Proof: Part $(ii) \Rightarrow (i)$ is clear because the sequence (x_n) is also a subsequence of itself.
For the Part $(i) \Rightarrow (ii)$, assume that $\lim x_n = a \in X$ exists. Let (x_{n_k}) be a subsequence of (x_n). We claim that $\lim x_{n_k} = a$. Let $\varepsilon > 0$. In fact, since $\lim x_n = a$, there is a positive integer N such that $d(a, x_n) < \varepsilon$ for all $n \ge N$. Notice that by the definition of a subsequence, there is a

positive integer K such that $n_k \geq N$ for all $k \geq K$. So, we see that $d(a, x_{n_k}) < \varepsilon$ for all $k \geq K$. Thus we have $\lim_{k\to\infty} x_{n_k} = a$.
Part $(ii) \Rightarrow (iii)$ is clear.
It remains to show Part $(iii) \Rightarrow (ii)$. Suppose that there are two subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ converge to distinct limits. Now put $k_1 := n_1$. Choose $m_{i'}$ such that $n_1 < m_{i'}$ and then put $k_2 := m_{i'}$. Then we choose n_i such that $k_2 < n_i$ and put k_3 for such n_i. To repeat the same step, we can get a subsequence $(x_{k_i})_{i=1}^{\infty}$ of (x_n) such that $x_{k_{2i}} = x_{n_{i'}}$ for some $n_{i'}$ and $x_{k_{2i-1}} = x_{m_{j'}}$ for some $m_{j'}$. Since by the assumption $\lim_i x_{n_i} \neq \lim_i x_{m_i}$, $\lim_i x_{k_i}$ does not exist which leads to a contradiction.
The proof is finished. $\square$

We now recall the following important theorem in $\mathbb{R}$ (see [1, Theorem 3.4.8]).

Theorem 3.2 Bolzano- Weierstrass Theorem *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Definition 3.3 X is said to be sequentially compact if for every sequence in X has a convergent subsequence.
In particular, a subset A of X is sequentially compact if every sequence has a convergent subsequence with the limit in A.

Example 3.4 (i) Every closed and bounded interval is sequentially compact.
In fact, if (x_n) is any sequence in a closed and bounded interval $[a, b]$, then (x_n) is bounded. Then by Bolzano-Weierstrass Theorem (see [1, Theorem 3.4.8]), (x_n) has a convergent subsequence (x_{n_k}). Notice that since $a \leq x_{n_k} \leq b$ for all k, then $a \leq \lim_k x_{n_k} \leq b$, and thus $\lim_k x_{n_k} \in [a, b]$. Therefore A is sequentially compact.

(ii) $(0, 1]$ is not sequentially compact. In fact, if we consider $x_n = 1/n$, then (x_n) is a sequence in $(0, 1]$ but it has no convergent subsequence with the limit sitting in $(0, 1]$.

Proposition 3.5 *If A is a sequentially compact subset of X, then A must be a closed and bounded subset of X.*

Proof: We first claim that A is bounded. Suppose not. We suppose that A is unbounded. If we fix an element $x_1 \in A$, then there is $x_2 \in A$ such that $d(x_1, x_2) > 1$. Using the unboundedness of A, we can find an element x_3 in A such that $d(x_3, x_k) > 1$ for $k = 1, 2$. To repeat the same step, we can find a sequence (x_n) in A such that $d(x_n, x_m) > 1$ for $n \neq m$. Thus A has no convergent subsequence. Thus A must be bounded
Finally, we show that A is closed in X. Let (x_n) be a sequence in A and it is convergent. It needs to show that $\lim_n x_n \in A$. Note that since A is compact, (x_n) has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Then by Proposition 3.1, we see that $\lim_n x_n = \lim_k x_{n_k} \in A$.
The proof is finished. $\square$

Corollary 3.6 *Let A be a subset of $\mathbb{R}$. Then A is sequentially compact if and only if A is a closed and bounded subset.*

Proof: The necessary part follows from Proposition 3.5 at once.
Now suppose that A is closed and bounded. Let (x_n) be a sequence in A and thus (x_n) is a

bounded sequence in $\mathbb{R}$. Then by the Bolzano- Weierstrass Theorem, (x_n) has a subsequence (x_{n_k}) which is convergent in $\mathbb{R}$. Since A is closed, $\lim_k x_{n_k} \in A$. Therefore, A is sequentially compact. □

Remark 3.7 From Corollary 3.6, we see that the converse of Proposition 3.5 holds when $X = \mathbb{R}$, but it does not hold in general. For example, if $X = \ell^\infty(\mathbb{N})$ and A is the closed unit ball in $\ell^\infty(\mathbb{N})$, that is $A := \{x \in \ell^\infty(\mathbb{N}) : \|x\|_\infty \leq 1\}$, then A is closed and bounded subset of $\ell^\infty(\mathbb{N})$ but it is not sequentially compact. Indeed, if we put $e_n := (e_{n,i})_{i=1}^\infty \in \ell^\infty(\mathbb{N})$, where $e_{n,i} = 1$ as $i = n$; otherwise, $e_{n,i} = 0$. Then (e_n) is a sequence in A but it has no convergent subsequence because $\|e_n - e_m\|_\infty = 2$ for $n \neq m$.

Definition 3.8 X is said to be compact if for any open cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of X, that is, each J_α is an open set and

$$X = \bigcup_{\alpha\in\Lambda} J_\alpha,$$

we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $X = J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

Remark 3.9 Notice that since for each open set V in $\mathbb{R}$ and for each element $x \in V$, we can find $r_x > 0$ such that $J_x := (x - r_x, x + r_x) \subseteq V$. So, we have $V = \bigcup_{x\in V} J_x$. Hence, in the Definition 3.8, those open sets J_α's can be replaced by open intervals.

Example 3.10 $(0, 1]$ is not compact. In fact, if we put $J_n = (1/n, 2)$ for $n = 2, 3...$, then $(0, 1] \subseteq \bigcup_{n=2}^\infty J_n$, but we cannot find finitely many $J_{n_1}, ..., J_{n_K}$ such that $(0, 1] \subseteq J_{n_1} \cup \cdots \cup J_{n_K}$. So $(0, 1]$ is not compact.

Proposition 3.11 *If X is compact, then it is sequentially compact.*

Proof: Suppose that X is compact but it is not sequentially compact. Then there is a sequence (x_n) in X such that (x_n) has no subsequent. Put $F = \{x_n : n = 1, 2, ...\}$. Then F is infinite and hence for each element $a \in X$, there is $\delta_a > 0$ such that $B(a, \delta_a) \cap F$ is finite. Indeed, if there is an element $a \in X$ such that $B(a, \delta) \cap F$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a. Let $J_a := B(a, \delta_a)$. On the other hand, we have $X = \bigcup_{a\in X} J_a$. Then by the compactness of X, we can find finitely many $a_1, ..., a_N$ such that $X = J_{a_1} \cup \cdots \cup J_{a_N}$. In particular, we have $F \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, F must be finite. This leads to a contradiction. Therefore, X is sequentially compact.
□

Remark 3.12 Indeed, we see that Proposition 3.11 holds for a general topological space from the proof above. The following Theorem 3.13 is an important feature of a metric space. We will show the case when $X = \mathbb{R}$ in the next section. The complete proof for the general metric spaces case is given in Section 5.

Theorem 3.13 *Let X be a metric space. Then X is sequentially compact if and only if it is compact.*

Proof: See Theorem 5.11 below (see also [2, Section 9.5, Theorem 16]). □

4 Sequentially Compact Sets and Compact Sets in $\mathbb{R}$

The following Lemma can be directly shown by the definition, so, the proof is omitted here.

Lemma 4.1 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) *A is closed.*

(ii) *For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.*

(iii) *If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.*

Before going to show Theorem 3.13 for the case of $\mathbb{R}$, let us first recall one of the important properties of real line.

Theorem 4.2 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

(i) : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) : $\lim_n (b_n - a_n) = 0$.

Then there is a unique real number ξ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. □

Theorem 4.3 (Heine-Borel Theorem) *Every closed and bounded interval $[a, b]$ is a compact set.*

Proof: Suppose that $[a, b]$ is not compact. Then there is an open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$ but it it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished. □

Theorem 4.4 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is compact.

(ii) A is sequentially compact.

(iii) A is closed and bounded.

Proof: *The result is shown by the following path* $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.
Part $(i) \Rightarrow (ii)$ *can be obtained by Proposition 3.11 immediately.*
Part $(ii) \Rightarrow (iii)$ *follows from Proposition 3.5 at once.*
It remains to show $(iii) \Rightarrow (i)$. *Suppose that* A *is closed and bounded. Then we can find a closed and bounded interval* $[a, b]$ *such that* $A \subseteq [a, b]$. *Now let* $\{J_\alpha\}_{\alpha \in \Lambda}$ *be an open intervals cover of* A. *Notice that for each element* $x \in [a, b] \setminus A$, *there is* $\delta_x > 0$ *such that* $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ *since* A *is closed. If we put* $I_x = (x - \delta_x, x + \delta_x)$ *for* $x \in [a, b] \setminus A$, *then we have*

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a,b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 4.3, we can find finitely many J_α*'s and* I_x*'s, say* $J_{\alpha_1}, ..., J_{\alpha_N}$ *and* $I_{x_1}, ..., I_{x_K}$, *such that* $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. *Note that* $I_x \cap A = \emptyset$ *for each* $x \in [a, b] \setminus A$ *by the choice of* I_x. *Therefore, we have* $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ *and hence* A *is compact.*
The proof is finished. □

5 Complete Metric Spaces

Let (X, d) be a metric space as before.

Definition 5.1 A sequence (x_n) in X is called a Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

Example 5.2 Let $e_n \in \ell^\infty(\mathbb{N})$ be defined as in Remark 3.7. Then (e_n) is not a Cauchy sequence.

Proposition 5.3 *Every convergent sequence is a Cauchy sequence.*

Proof: Let (x_n) be a convergent sequence in X. Suppose that $\lim_n x_n = v \in X$. Then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(v, x_n) < \varepsilon$ for all $n \geq N$. Thus for any $m, n \geq N$, we see that $d(x_m, x_n) \leq d(x_m, v) + d(v, x_n) < 2\varepsilon$. Thus (x_n) is a Cauchy sequence. □

Remark 5.4 The converse of Proposition 5.3 does not hold in general. For example, if we consider $X = (0, 1]$ and $x_n = 1/n$, then (x_n) is a Cauchy sequence but it is not convergent in $(0, 1]$.

The following definition is one of important concepts in mathematics world.

Definition 5.5 X is said to be complete if every Cauchy sequence in X is convergent.

The following result is a very important motivation of the definition of completeness.

Theorem 5.6 $\mathbb{R}$ *is complete.*

Proof: Let (x_n) be a Cauchy sequence in $\mathbb{R}$. We first claim that (x_n) must be bounded. Indeed, by the definition of a Cauchy sequence, if we consider $\varepsilon = 1$, then there is a positive integer N such that $|x_m - x_N| < 1$ for all $m \geq N$ and thus we have $|x_m| < 1 + |x_N|$ for all $m \geq N$. So, if we let $M = \max(|x_1|, ..., |x_{N-1}|, |x_N| + 1)$, then we have $|x_n| \leq M$ for all n. Hence (x_n) is bounded.
So, we can now apply the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Let $L := \lim_k x_{n_k}$. We are going to show that $L = \lim_n x_n$.
Let $\varepsilon > 0$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$. On the other hand, since $\lim_k x_{n_k} = L$, we can find a positive integer K so that $|L - x_{n_k}| < \varepsilon$ for all $k \geq K$. Now if we choose $r \geq K$ such that $n_r \geq N$, then for any $n \geq N$, we have $|x_n - L| \leq |x_n - x_{n_r}| + |x_{n_r} - L| < 2\varepsilon$. Thus (x_n) is convergent with $\lim_n x_n = L$.
The proof is finished. □

Example 5.7 (i) $\ell^\infty(\mathbb{N}) := \{(x_i)_{i=1}^\infty : \sup_i |x_i| < \infty\}$ is complete under the sup norm $\|\cdot\|_\infty$. In fact, notice that if $(\mathbf{x_n})$ is Cauchy sequence in ℓ^∞ and if we let $\mathbf{x_n} = (x_{n,i})_{i=1}^\infty$, then for each $i = 1, 2...$, $(x_{n,i})_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Thus $\lim_n x_{n,i}$ exists in $\mathbb{R}$ for each i. Write $\xi_i := \lim_n x_{n,i} \in \mathbb{R}$ and $\xi := (\xi_i)$. We are now going to show that $\xi \in \ell^\infty$ and $\lim_n \|\xi - x_n\|_\infty = 0$.
Notice that since (x_n) is a Cauchy sequence in ℓ^∞, so, for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\|_\infty < \varepsilon$ for all $m, n \geq N$ and hence we have

$$|x_{n,i} - x_{m,i}| \leq \sup_k |x_{n,k} - x_{m,k}| = \|\mathbf{x_n} - \mathbf{x_m}\|_\infty < \varepsilon$$

for all $m, n \geq N$ and for all $i = 1, 2....$ So if we fix i and $m \geq N$ and taking $n \to \infty$, then we have $|\xi_i - x_{m,i}| < \varepsilon$ and hence $\|\xi - \mathbf{x_m}\|_\infty < \varepsilon$ for $m \geq N$. From this we see that $\lim_m \|\xi - \mathbf{x_m}\|_\infty = 0$ and thus $\xi \in \ell^\infty$ because ℓ^∞ is a vector space.

(ii) $c_0(\mathbb{N})$ is complete under the sup-norm. In fact every closed subset of a compete metric space must be complete **(why?)**. Since c_0 is closed in ℓ^∞, c_0 is complete.

(iii) $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$ all are complete metric spaces under the ℓ^p-norm.

(iv) $C[a, b] := \{f : [a, b] \to \mathbb{R} : \text{f is continuous}\}$ is complete under the sup-norm.

Proposition 5.8 *Let (F_n) be a sequence of closed and bounded non-empty subsets of a complete metric space X. For each n, put $diam(F_n) := \sup\{d(x, y) : x, y \in F_n\}$ (the diameter of F_n). Suppose that it satisfies the following conditions.*

(a) $F_1 \supseteq F_2 \supseteq F_3 \cdots\cdots$.

(b) $\lim_n diam(F_n) = 0$.

If X is complete, then there is a unique element $\xi \in X$ such that $\bigcap_n F_n = \{\xi\}$.

Proof: For each F_n, we take an element x_n in F_n. Then by the condition of (a) and (b) above, (x_n) forms a Cauchy sequence in X. Since X is complete, $\xi := \lim x_n$ exists in X. Note that $\xi \in F_n$ for all n because each F_n is closed and $F_m \supseteq F_{m+1} \supseteq \cdots$ for all m. So, $\xi \in \bigcap_n F_n$.
On the other hand, the condition (b) implies that the intersection $\bigcap_n F_n$ contains at most one element. The proof is finished. □

Remark 5.9 The assumption of completeness of X in Proposition 5.8 is essential. For example, if we consider $X = (0, 1]$ and $F_n = (0, \frac{1}{n+1}]$ for $n = 1, 2...$, then F_n's satisfies the conditions (a) and (b) above but $\bigcap_n F_n = \emptyset$.

Definition 5.10 X is said to be totally bounded if for any $r > 0$, there exists finitely many open balls of radius r, say $B_1, ..., B_N$ such that $X = B_1 \cup \cdots \cup B_N$.

The following can be viewed as the generalization of the real case (see Theorem 5.6).

Theorem 5.11 *The following statements are equivalent.*

(i) X is compact.

(ii) X is sequentially compact.

(iii) X is complete and totally bounded.

Proof: Part $(i) \Rightarrow (ii)$ has been shown in Proposition 3.11.
For Part $(ii) \Rightarrow (iii)$, assume that X is sequentially compact. We first claim that X is complete. Let (x_n) be a Cauchy sequence in X. Notice that (x_n) has a convergent subsequence (x_{n_k}) from the assumption. Let $\lim_k x_{n_k} = v \in X$. Using the same argument as in the proof of Theorem 5.6, we see that $v = \lim_n x_n$ and hence X is complete.
Secondly, we show that X is totally bounded. Suppose not. Then there is $r > 0$ such that X cannot be covered by finitely many open balls of radius r. Fix $x_1 \in X$. Then there is $x_2 \in X$ with $d(x_2, x_1) \geq r$. Similarly, we can find $x_3 \in X$ such that $d(x_3, x_k) \geq r$ for $k = 1, 2$ because the choice of r. To repeat the same argument, we have a sequence (x_n) in X such that $d(x_n, x_m) \geq r$ for all $n \neq m$. Therefore, (x_n) has no convergent subsequence. It leads to a contradiction and hence X must be totally bounded.
It remains to show $(iii) \Rightarrow (i)$. Assume that X is complete and totally bounded.
Suppose that X is not compact. Then there is open cover of X, says $\mathcal{J} := \{J_i\}_{i \in I}$, which has no finite subcover of X.
Since X is totally bounded by the assumption, then there are finitely many open balls $B_1, ..., B_m$ and each ball has radius 1 such that $X = B_1 \cup \cdots \cup B_m$. Since $\mathcal{J}$ has no finite subcover, there must exist some B_k which cannot be covered by finitely J_i's. Let B_1 be such open ball. Put $F_1 := \overline{B_1}$ and hence F_1 also cannot be covered by finitely many J_i's. Using totally boundedness of X again, we can find finitely many open balls $D_1, ..., D_l$ and each has radius $1/2$ such that $F_1 \subseteq D_1 \cup \cdots \cup D_l$ and $D_i \cap F_1 \neq \emptyset$ for all $i = 1, .., l$. Since F_1 cannot be covered by finitely many J_i's, there must exist some D_j such that $F_1 \cap D_j$ shares the same property. Put $F_2 := \overline{F_1 \cap D_j}$. To repeat the same step, we can get a sequence of closed and bounded subsets (F_n) of X which has the following properties.

(a) $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots\cdots$.

(b) $diam(F_n) \to 0$ as $n \to \infty$.

(c) Each F_n cannot be covered by finitely many J_i's.

By using Proposition 5.8, we have $\bigcap_n F_n = \{\xi\}$ for some element $\xi \in X$. On the other hand, we have $\xi \in J_{i_0}$ for some $J_{i_0} \in \mathcal{J}$. Since J_{i_0} is open, there is $r > 0$ such that $B(\xi, r) \subseteq J_{i_0}$. It is because $\lim_n diam(F_n) = 0$, we can find F_N such that $diam(F_N) < r$. Since $\xi \in F_N$, we have $F_N \subseteq B(\xi, r) \subseteq J_{i_0}$ which contradicts to the property (c) above.
The proof is finished. □

Exercise 5.12 Let A be a subset of X.

(i) Show that if X is complete, then A is complete if and only if A is closed in X.

(ii) Show that if A is complete, then A is closed in X.

6 Continuous mappings

Throughout this section, let (X, d) and (Y, ρ) be metric spaces.

Definition 6.1 Let $f : X \to Y$ be a function from X into Y. We say that f is continuous at a point $c \in X$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, y) < \delta$.
Furthermore, f is said to be continuous on A if f is continuous at every point in X.

Remark 6.2 It is clear that f is continuous at $c \in X$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon))$.

Proposition 6.3 *With the notation as above, we have*

(i) f is continuous at some $c \in X$ if and only if for any sequence $(x_n) \in X$ with $\lim x_n = c$ *implies* $\lim f(x_n) = f(c)$.

(ii) The following statements are equivalent.

(ii.a) f is continuous on X.

(ii.b) $f^{-1}(W) := \{x \in X : f(x) \in W\}$ is open in X for any open subset W of Y.

(ii.c) $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is closed in X for any closed subset F of Y.

Proof: Part (i):
Suppose that f is continuous at c. Let (x_n) be a sequence in X with $\lim x_n = c$. We claim that $\lim f(x_n) = f(c)$. In fact, let $\varepsilon > 0$, then there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, c) < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $d(x_n, c) < \delta$ for $n \geq N$ and hence $\rho(f(x_n), f(c)) < \varepsilon$ for all $n \geq N$. Thus $\lim f(x_n) = f(c)$.
For the converse, suppose that f is not continuous at c. Then we can find $\varepsilon > 0$ such that for any n, there is $x_n \in X$ with $d(x_n, c) < 1/n$ but $\rho(f(x_n), f(c)) \geq \varepsilon$. So, if f is not continuous

at c, then there is a sequence (x_n) in X with $\lim x_n = c$ but $(f(x_n))$ does not converge to $f(c)$.
Part $(iia) \Leftrightarrow (iib)$:
Suppose that f is continuous on X. Let W be an open subset of Y and $c \in f^{-1}(W)$. Since W is open in Y and $f(c) \in W$, there is $\varepsilon > 0$ such that $B(f(c), \varepsilon) \subseteq W$. Since f is continuous at c, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon)) \subseteq f^{-1}(W)$. So $f^{-1}(W)$ is open in X.
It remains to show that the converse of Part (ii). Let $c \in X$. Let $\varepsilon > 0$. Put $W := B(f(c), \varepsilon)$. Then W is an open subset of Y and thus $c \in f^{-1}(W)$ and $f^{-1}(W)$ is open in X. Therefore, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(W)$. So, f is continuous at c.
Finally, the last equivalent assertion $(ii.b) \Leftrightarrow (ii.c)$ is clearly from the fact that a subset of a metric space is closed if and only if its complement is open in the given metric space (see Proposition 2.16 (i)).
The proof is complete. □

Corollary 6.4 *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between metric spaces. Then the composition $g \circ f : X \to Z$ is also continuous on X.*

Proof: It is clear from Proposition 6.3 at once. □

Definition 6.5 A bijection $f : X \to Y$ is said to be a homeomorphism if f and its inverse f^{-1} both are continuous. In this case, X is said to be homeomorphic to Y.

Proposition 6.6 *If $f : X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is also a compact subset of Y. Consequently, if f is a continuous bijection, then f must be a homeomorphism, that is, the inverse map $f^{-1} : Y \to X$ is automatically continuous.*

Proof: Let $\{V_i\}_{i \in I}$ be an open cover of $f(X)$, that is, each V_i is an open subset of Y and $f(X) \subseteq \bigcup_{i \in I} V_i$. Hence $\{f^{-1}(V_i)\}_{i \in I}$ is also an open cover of X by Proposition 6.3. So by the compactness of X, there are finitely many $i_1, .., i_N \in I$ such that $X = f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_N})$. This gives $f(A)$ is covered by $V_{i_1}, ..., , V_{i_N}$. Thus $f(A)$ is compact.
For showing the inverse $f^{-1} : Y \to X$, by Proposition 6.3, it needs to show that $f(F) = (f^{-1})^{-1}(F)$ is a closed subset of Y for every closed subset F of X. In fact, it is easy to see that every closed subset of a compact metric space must be compact and every compact subset of a metric space is also closed. Hence F is a compact subset of X and thus $f(F)$ is compact by above. So $f(F)$ is a closed subset of Y as desired. The proof is finished. □

Definition 6.7 We say that two metrics d_1 and d_2 on a set X are equivalent if there are positive constants c, c' such that $c'd_1(x, y) \le d_2(x, y) \le cd_1(x, y)$ for all $x, y \in X$.

Example 6.8 Let $X = (0, 1)$ and d be the usual metric on X, that is $d(x, y) := |x - y|$. Define a metric on X by $\rho(x, y) := \frac{|x-y|}{1+|x-y|}$ for $x, y \in (0, 1)$. Then the metrics d and ρ are equivalent on $(0, 1)$. In fact, one can directly check that we have $\rho(x, y) \le d(x, y) \le 2\rho(x, y)$ for all $x, y \in (0, 1)$.

Proposition 6.9 *Let d_1 and d_2 be the metrics on X. If d_1 and d_2 are equivalent, then the identity map $I : (X, d_1) \to (X, d_2)$ is a homeomorphism.*

Proof: It clearly follows from Proposition 6.3. □

Remark 6.10 (i) The converse of Proposition 6.9 does not hold. For example, let $X = \mathbb{R}$ and d the usual metric. Let ρ be given as in Proposition 6.9. Then for a sequence (x_n) and an element x in $\mathbb{R}$, we see that $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. So, the identity $I : (X, d) \to (X, \rho)$ is a homeomorphism. However, if $X = \mathbb{R}$, then the usual metric d is not equivalent to the metric ρ defined above. In fact, although we always have $\rho(x, y) \le d(x, y)$ for all $x, y \in \mathbb{R}$, it is impossible to find a positive constant c such that $d(x, y) \le c\rho(x, y)$ for all $x, y \in \mathbb{R}$. Notice that if there is such c, then we have $|x - y| = d(x, y) \le c - 1$ for all $x \ne y$ in $\mathbb{R}$. It is absurd.

(ii) The completeness of metric spaces are not preserved under homeomorphisms. For example, consider $X = \mathbb{R}$. Let $d_1(x, y) := |x - y|$ and $d_2(x, y) := |e^{-x} - e^{-y}|$ for x, y in $\mathbb{R}$. Then the identity map $I : (X,\ d_1) \to (X,\ d_2)$ is a homeomorphism **(check)!** and $(X,\ d_1)$ is complete. However, $(X,\ d_2)$ is not complete. In fact, if we let $x_n = n$ for $n = 1, 2...$, then (x_n) is Cauchy but not convergent in $\mathbb{R}$ with respect to the metric d_2.

Definition 6.11 A mapping $f : (X, d) \to (Y, \rho)$ is called a contraction if there is $0 < r < 1$ such that $\rho(f(x), f(x')) \le rd(x, x')$ for all $x, x' \in X$.

Remark 6.12 It is clear that every contraction must be continuous.

Example 6.13 (i) Define $f : (1, \infty) \to (1, \infty)$ by $f(x) := \sqrt{x}$. Then f is a contraction since we always have $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all $x, y \in (1, \infty)$. Indeed, for any $x, y \in (1, \infty)$ with $x < y$, then by the Mean Value Theorem, there is $c \in [x, y]$ such that $f(x) - f(y) = f'(c)(x - y)$. Notice that $f'(c) = \frac{1}{2\sqrt{c}} \le \frac{1}{2}$.

Proposition 6.14 *Let (X, d) be a complete metric space. If $f : X \to X$ is a contraction, then there is a fixed point for f, that is, there is $c \in X$ such that $f(c) = c$.*

Proof: Let $0 < r < 1$ be a contraction ratio for f, that is, $d(f(x), f(y) \le rd(x, y)$ for all $x, y \in X$. Fix $x_1 \in X$. Put $x_{n+1} = f(x_n)$, for $n = 1, 2, ...$
We first claim that (x_n) is a Cauchy sequence in X. In fact, notice that we have $d(x_{n+2}, x_{n+1}) = d(f(x_{n+1}), f(x_n)) \le rd(x_{n+1}, x_n)$ for all $n = 1, 2....$ So, we have

$$d(x_{n+1}, x_n) \le r^{n-1} d(x_2, x_1)$$

for all $n = 1, 2....$ From this, we have

$$d(x_{n+p}, x_n) \le \sum_{n \le k \le n+p-1} d(x_{k+1}, x_k) \le \sum_{n \le k \le n+p-1} r^k d(x_2, x_1) \tag{6.1}$$

for any $n, p = 1, 2....$ On the other hand, since $0 < r < 1$, we have $\sum_{k=1}^{\infty} r^k < \infty$ and hence, for any $\varepsilon > 0$, there is a positive integer N such that $\sum_{k=n}^{\infty} r^k < \varepsilon$ for all $n \ge N$. So, by the Eq 6.1 above, we see that (x_n) is a Cauchy sequence in X. This implies that $\lim x_n = c$ exists in X because X is complete. Since f is continuous and $x_{n+1} = f(x_n)$, the result follows from

$$c = \lim x_{n+1} = \lim f(x_n) = f(c).$$

The proof is finished. □

Remark 6.15 The Proposition 6.14 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, then it is clear that $|f(x) - f(y)| = |x - y|$ and f has no fixed point in $\mathbb{R}$.

Exercise 6.16 A function $g : (X, d) \to (Y, \rho)$ is called a Lipschitz function if there is a $C > 0$ such that $\rho(g(x), g(x')) \leq Cd(x, x')$ for all $x, x' \in X$. Now let $A \subseteq X$ be a non-empty subset and assume that $f : A \to Y$ is a Lipschitz function.

(i) Show that f is continuous on A.

(ii) Show that if (x_n) is a Cauchy sequence in A, then $f(x_n)$ is a Cauchy sequence in Y.

(iii) Show that if Y is complete, then there is a unique continuous mapping $F : \overline{A} \to Y$ such that $F(x) = f(x)$ for all $x \in A$.

Answer:
Part (i) and (ii) are clearly shown by the definition of Lipschitz functions.
The proof of Part (iii) is divided by the following several claims.
Claim 1. If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.
Claim 2. If (x_n) and (y_n) both are convergent sequences in A and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.
By **Claim 1**, $L := \lim f(x_n)$ and $L' = \lim f(y_n)$ both exist in Y. For any $\varepsilon > 0$, let $\delta > 0$ be found as in **Claim 1**. Since $\lim x_n = \lim y_n$, there is $N \in \mathbb{N}$ such that $d(x_n, y_n) < \delta$ for all $n \geq N$ and hence, we have $\rho(f(x_n), f(y_n)) < \varepsilon$ for all $n \geq N$. Taking $n \to \infty$, we see that $\rho(L, L') \leq \varepsilon$ for all $\varepsilon > 0$. So $L = L'$. **Claim 2** follows.
Recall that an element $x \in \overline{A}$ if and only if there is a sequence (x_n) in A converging to x.
Now for each $x \in \overline{A}$, we define

$$F(x) := \lim f(x_n)$$

if (x_n) is a sequence in A with $\lim x_n = x$. It follows from **Claim 1** and **Claim 2** that F is a well defined function defined on $\overline{A}$ and $F(x) = f(x)$ for all $x \in A$.
So, it remains to show that F is continuous. Then F is a continuous extension of f to $\overline{A}$ as desired.
Now suppose that F is not continuous at some point $z \in \overline{A}$. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there is $x \in \overline{A}$ satisfying $d(x, z) < \delta$ but $\rho(F(x), F(z)) \geq \varepsilon$. Notice that for any $\delta > 0$ and if $d(x, z) < \delta$ for some $x \in \overline{A}$, then we can choose a sequence (x_i) in A such that $\lim x_i = x$. Therefore, we have $d(x_i, z) < \delta$ and $\rho(f(x_i), F(z)) \geq \varepsilon/2$ for any i large enough. Therefore, for any $\delta > 0$, we can find an element $x \in A$ with $d(x, z) < \delta$ but $\rho(f(x), F(z)) \geq \varepsilon/2$. Now consider $\delta = 1/n$ for $n = 1, 2....$ This yields a sequence (x_n) in A which converges to z but $\rho(f(x_n), F(z)) \geq \varepsilon/2$ for all n. However, we have $\lim f(x_n) = F(z)$ by the definition of F which leads to a contradiction. Thus F is continuous on $\overline{A}$.
Finally the uniqueness of such continuous extension is clear.
The proof is finished.

Remark 6.17 In general, the continuous extension of a continuous function may not exist. For example, let $X = Y = \mathbb{R}$ and $A = (0, 1]$. If we consider $f(x) = 1/x$ for $x \in A$, then f does not have continuous extension to $\overline{A} = [0, 1]$. In fact, if such continuous extension F exists on $[0, 1]$, then F must be bounded on $[0, 1]$, in particular, it is bounded on $(0, 1]$ and hence, $F(x) = f(x) = 1/x$ is bounded on $(0, 1]$. It leads to a contradiction.

Definition 6.18 A mapping $f : (X, d) \to (Y, \rho)$ is said to be uniformly continuous on X if for any $\varepsilon > 0$, there is $\delta > 0$, such that $\rho(f(x), f(x')) < \varepsilon$, whenever, $d(x, x') < \delta$.

Proposition 6.19 *If $f : (X, d) \to (Y, \rho)$ is continuous and X is compact, then f is uniformly continuous on X.*

Proof: **Compactness argument**:
Let $\varepsilon > 0$. Since f is continuous on A, then for each $x \in X$, there is $\delta_x > 0$, such that $\rho(f(y), f(x)) < \varepsilon$ whenever $y \in X$ with $d(x, y) < \delta_x$. Now for each $x \in X$, set $J_x = B(x, \frac{\delta_x}{2})$. Then $X \subseteq \bigcup_{x \in X} J_x$. By the compactness of X, there are finitely many $x_1, ..., x_N \in X$ such that $X = J_{x_1} \cup \cdots \cup J_{x_N}$. Now take $0 < \delta < \min(\frac{\delta_{x_1}}{2}, ..., \frac{\delta_{x_N}}{2})$. Now for $x, y \in X$ with $d(x, y) < \delta$, then $x \in J_{x_k}$ for some $k = 1, .., N$, from this it follows that $d(x, x_k) < \frac{\delta_{x_k}}{2}$ and $d(y, x_k) \leq d(y, x) + d(x, x_k) \leq \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}$. So for the choice of δ_{x_k}, we have $\rho(f(y), f(x_k)) < \varepsilon$ and $\rho(f(x), f(x_k)) < \varepsilon$. Thus we have shown that $\rho(f(x), f(y)) < 2\varepsilon$ whenever $x, y \in X$ with $d(x, y) < \delta$. The proof is finished.
Sequentially compactness argument:
Suppose that f is not uniformly continuous on X. Then there is $\varepsilon > 0$ such that for each $n = 1, 2, ..$, we can find x_n and y_n in X with $d(x_n, y_n) < 1/n$ but $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Notice that by the sequentially compactness of X, (x_n) has a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in X$. Now applying sequentially compactness of X for the sequence (y_{n_k}), then (y_{n_k}) contains a convergent subsequence $(y_{n_{k_j}})$ such that $b := \lim_j y_{n_{k_j}} \in X$. On the other hand, we also have $\lim_j x_{n_{k_j}} = a$. Since $d(x_{n_{k_j}}, y_{n_{k_j}}) < 1/n_{k_j}$ for all j, we see that $a = b$. This implies that $\lim_j f(x_{n_{k_j}}) = f(a) = f(b) = \lim_j f(y_{n_{k_j}})$. This leads to a contradiction since we always have $\rho(f(x_{n_{k_j}}), f(y_{n_{k_j}})) \geq \varepsilon > 0$ for all j by the choice of x_n and y_n above. The proof is finished. □

Proposition 6.20 *Assume that X and Y are complete. Let A be a subset of X and $f : A \to Y$ a continuous function. If A is totally bounded, then the following statements are equivalent.*

(i): f is uniformly continuous on A.

(ii): There is a unique continuous function F defined on the closure $\overline{A}$ such that $F(x) = f(x)$ for all $x \in A$.

Proof: For the Part $(ii) \Rightarrow (i)$, we first notice that $\overline{A}$ is also totally bounded while A is totally bounded. Indeed, for any $r > 0$, we can find finitely many element $x_1, .., x_N$ in A such that $A \subseteq B(x_1, r/2) \cup \cdots \cup B(x_N, r/2)$. Now for any $z \in \overline{A}$, we have $B(z, r/2) \cap A \neq \emptyset$ and hence, $B(z, r/2) \cap B(x_k, r/2) \neq \emptyset$ for some k. It implies that $z \in B(x_k, r)$. So, $A \subseteq B(x_1, r) \cup \cdots \cup B(x_N, r)$. Therefore, $\overline{A}$ is totally bounded too. Then by Theorem 5.11, $\overline{A}$ is compact since X is complete. Thus, the implication $(ii) \Rightarrow (i)$ follows from Proposition 6.19 at once.
The proof of Part $(i) \Rightarrow (ii)$ is exactly the same in Exercise 6.16. Assume that f is uniformly continuous on A.
We first notice that if (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.
It needs to show that $(f(x_n))$ is a Cauchy sequence because Y is complete. Indeed, let $\varepsilon > 0$. Then by the uniform continuity of f on A, there is $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever

$x, y \in A$ with $d(x, y) < \delta$. Notice that (x_n) is a Cauchy sequence since it is convergent. Thus, there is a positive integer N such that $d(x_m, x_n) < \delta$ for all $m, n \geq N$. This implies that $\rho(f(x_m), f(x_n)) < \varepsilon$ for all $m, n \geq N$ and hence, $\lim f(x_n)$ exists in Y.
Then the rest of the proof follows from Exercise 6.16 at once. □

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (*4th ed*), Wiley, (2011).

[2] H. Royden and P. Fitzpatrick, Real Analysis, (*4th ed*), Pearson, (2010).